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Auction Theory–Past Exam Papers

Contents

1 Formulae

$$F(x) = \text{Prob}[X < x] \qquad f(x) = F'(x)$$

$$\int_{x_0}^{x^1} f(x)\,dx = \text{Prob}[x_0 < x < x_1]$$

$$E(X|X < x) = \frac{1}{F(x)} \int_0^x t\, f(t)\,dt$$

$$\lambda(x) \equiv \frac{f(x)}{1-F(x)} \qquad \sigma(x) \equiv \frac{f(x)}{F(x)}$$

$$F(x) = 1 - \exp\left(-\int_0^x \lambda(t)\,dt\right) = $$
$$= \exp\left(-\int_x^\omega \sigma(t)\,dt\right)$$

$$F_1^{(n)}(y) = F(y)^n$$
$$f_1^{(n)}(y) = n \cdot F(y)^{n-1} \cdot f(y)$$
$$f_2^{(n)}(y) = f(y) \cdot F(y)^{n-2} \cdot (n-1) \cdot n \cdot \Big(1 - F(y)\Big)$$
$$f_2^{(n)}(y) = f_1^{(n-1)} \cdot n \cdot \Big(1 - F(y)\Big)$$

$$f_{1,2}^{(n)}(y_1, y_2) = $$
$$= \begin{cases} F(y_2)^{n-2} \cdot f(y_1) \cdot f(y_2) \cdot n \cdot (n-1) & \text{if } y_1 \geqslant y_2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_2^{(n)}\left(z \middle| Y_1^{(n)} = y\right) = f_1^{(n-1)}\left(z \middle| Y_1^{(n-1)} \leqslant y\right)$$

$$m^I(x) = G(x) \cdot \beta^I(x)$$
$$m^{II}(x) = G(x) \cdot E\left[Y_2^{(n)} | Y_1^{(n)} = X\right]$$
$$\beta^I(x) = E\left[Y_1^{(n-1)} | Y_1^{(n-1)} \leqslant x\right] = x - \int_0^x \frac{G(y)}{G(x)}\,dy$$
$$\beta^I(x,r) = \frac{1}{G(x)}\left(r \cdot G(r) + \int_r^x y \cdot g(y)\,dy\right)$$
$$m^I(x,r) = r \cdot G(r) + \int_r^x y \cdot g(y)\,dy$$

2 Exam, 6. 3. 2023

The exam was set as an open book exam at the University of Jena. Participants had 60 minutes to answer all questions. The exam sheet also contained the formulae from Section 1.

Please solve as many problems as you can and please label your answers (i.e. make clear to which problem your answer belongs). If you have not answered all questions at the end, don't worry. Submit the answers you have so far as a PDF on the moodle exam server. Good luck!

1. Consider an auction for a single object with four bidders $i \in \{1, 2, 3, 4\}$ where valuations x_i are private and are distributed according to

$$F(x_i) = \begin{cases} 0 & \text{if } x_i < 0 \\ x_i/4 & \text{if } 0 \leqslant x_i \leqslant 4 \\ 1 & \text{otherwise} \end{cases}$$

   for all bidders.

   a) Consider a bidder with valuation $x_i \in [0, 4]$ in this auction. How probable is it that all other bidders have a smaller valuation than this bidder?
   b) What is the symmetric equilibrium bidding function in a first-price auction?
   c) What is the expected payment ex-interim (after learning x_i) for bidder i?
   d) What is this bidder's ex-interim expected net profit from participating in the auction?
   e) What is the expected payment ex-ante (before learning x_i)?
   f) What is this bidder's ex-ante expected net profit from participating in the auction?
   g) What is the seller's expected revenue?
   h) What is the symmetric equilibrium bidding function in a second-price auction?
   i) What is the symmetric equilibrium bidding function in an all-pay auction?

2. Now consider, within the context of problem 1, a first-price auction with a reserve price r. (This problem is a bit more involved, so you might want to solve this problem at the end of the exam, if there is still time left.)

   a) What is now the symmetric equilibrium bidding function?
   b) What is now the expected payment ex-interim?
   c) What is now the expected payment ex-ante?
   d) Assume the seller has an own valuation for the object of $x_0 = 3$. What is the seller's profit maximising reserve price r?
   e) Assume the seller values the object at $x_0 = 1$. What is the revenue-maximising reserve-price?

3. Consider an auction with three bidders $i \in \{1, 2, 3\}$ and private valuations x_i which are distributed independently and uniformly over the interval $[0, 2]$. Bidder 1 knows that bidders 2 and 3 bid according to $\beta_2(x_2) = x_2/2$ and $\beta_3(x_3) = x_3/2$.

   a) Assume bidder 1 makes a bid b. Conditional on this bid, what is the probability that bidder 1 wins the auction?
   b) In a first-price auction, what is the best reply of bidder 1 to the above bidding strategies of bidder 2 and 3?
   c) In a second-price auction, what is the best reply of bidder 1?
   d) In an all-pay auction, what is the best reply of bidder 1?

4. Consider an auction with three bidders $i \in \{1, 2, 3\}$ and common valuations. Before the auction starts, each bidder i obtains a signal s_i. These signals are drawn independently from a uniform distribution over $[0, 1]$. Bidder i's valuation is $x_i = s_1 + s_2 + s_3$.

   The format of the auction is the variant of the English auction that we have discussed in the lecture: At the beginning of the auction all bidders participate. The price rises gradually. Bidders can leave the auction at any time. All bidders see when one bidder leaves the auction. We call b_1 the price when the first bidder leaves. We call b_2 the price when the second bidder leaves. Once two bidders have left, the remaining bidder wins the object and pays b_2. Bidders know their own signal, but they don't know the signals of their competitors. They see, however, when their competitors leave. We assume that bidders follow a symmetric equilibrium strategy.

a) Before a bidder knows their own signal (ex ante), how probable is it that this bidder wins the auction?

b) What is the expected valuation ex ante (i.e. the valuation before a bidder learns their signal)?

c) Once bidder i knows their own signal s_i, but before the auction has started, how probable is it now that bidder i wins the auction?

d) Once bidder i knows their own signal s_i, but before the auction has started, what is bidder i's ex-interim valuation of the object?

e) While all bidders still participate in the auction, what is the bid in the symmetric equilibrium? This bid can only condition on the own signal, s_i.

f) After one bidder has left the auction at a price of b_1, what is the bid in the symmetric equilibrium. This bid can be conditional on the own signal s_i of a remaining bidder and on the price b_1 when the first bidder left.

g) After two bidders have left at prices b_1 and b_2, assume that bidder i has won the auction. What is bidder i's ex-post valuation of the object?

h) What is bidder i's ex-post gain from the auction?

i) What is bidder i's ex-interim gain from participating in the auction? This is the situation after the bidder learns the own signal s_i, but before bidding starts, and before bidder i knows who wins the auction.

j) What is bidder i's ex-ante gain from participating in the auction? This is before bidder i learns their own signal.

3 Exam, 27. 7. 2019

The exam was set at the University of Jena. Participants had 60 minutes to answer all questions. The exam sheet also contained the formulae from Section 1.

Explain your answers briefly.

1. Consider an auction with two bidders and valuations which are distributed independently and uniformly over the interval $[0, 1]$. Bidder 1 knows that bidder 2 uses the bidding function $b_2(x_2) = \sqrt{x_2}$.

    a) What is a best reply of bidder 1 in a first price auction?

    b) What is the expected payment of bidder 1 with valuation x in a first price auction?

    c) What is the ex-ante expected payment (i.e. before the own valuation is known) of a bidder 1 in a first price auction?

    d) What is the best reply of bidder 1 in a second price auction?

    e) What is the expected payment of a bidder 1 with valuation x in a second price auction?

    f) What is the best reply of bidder 1 in an all-pay auction?

2. Consider an auction for a single object with two bidders where valuations are distributed according to

$$F(x) = \begin{cases} 0 & x < 0 \\ x^3 & x \in [0, 1] \\ 1 & \text{otherwise} \end{cases}$$

for both bidders.

    a) What is the symmetric equilibrium bidding function in the first price auction?

    b) What is the expected payment ex-interim (i.e. once a bidders knows the own valuation) in the equilibrium of the first price auction for each bidder?

    c) What is for each bidder the expected payment ex-ante (i.e. before a bidder knows the own valuation) in the equilibrium of the first price auction?

    d) What is in this case the seller's expected revenue?

    e) What is the symmetric equilibrium bid in the all-pay auction?

    f) Now assume the seller can set a reserve price r. Bidders still have valuations as above. What is the expected payment in the first price auction with a reserve price r for a bidder with valuation $x > r$?

g) What is in this situation the symmetric equilibrium bidding function for a bidder with valuation $x > r$?

h) What is the seller's expected revenue?

i) What is the revenue maximising reserve price?

4 Exam, 26. 2. 2013

The exam was set at the University of Jena. Participants had 60 minutes to answer all questions. The exam sheet also contained the formulae from Section 1.

1. Consider an auction for a single object with two bidders where $\omega = 1$ and valuations are distributed according to

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & x \in [0, 1] \\ 1 & \text{otherwise} \end{cases}$$

for both bidders. What is the symmetric equilibrium bidding function?

2. What is the expected payment ex-interim?

3. What is the expected payment ex-ante?

4. What is the seller's expected revenue?

5. Now assume the seller can set a reserve price r. The two bidders still have valuations distributed as above. What is the symmetric equilibrium bidding function?

6. What is the expected payment ex-interim?

7. What is the expected payment ex-ante?

8. What is the seller's expected revenue if the seller has an own valuation of x_0 for the object? (the expected revenue should take into account this own valuation in case the object remains unsold.)

9. Assume $x_0 = 1/4$. What is the seller's profit maximising r?

10. So far we have (in the lecture and also in this exam) assumed that bidders maximise expected payoffs, i.e. that they are risk-neutral. Let us drop this assumption and instead assume that bidders maximise the expectation of a utility function $u(x) = x^\rho$ where x is the payoff (i.e. the difference between monetary valuation and payment) and $\rho > 0$ is a parameter that characterises attitudes towards risk. Return to problem (1). Can you use the revenue-equivalence principle to find an equilibrium here, now under the assumption that expected utility, and not expected payment is maximised? Explain your answer!

11. Try to formulate a condition for $\beta'(x)$ in equilibrium.

12. Solve this condition for $\beta(x)$.

5 Exam, 21. 7. 2009

The exam was set at the University of Jena. Participants had 60 minutes to answer all questions. The exam sheet also contained the formulae from Section 1.

1. Consider an auction with two bidders. Valuations x are uniformly and independently distributed for both bidders over the interval $[0, 1]$. Bidder 1 knows that bidder 2 always follows the bidding stratagy $b_2(x) = x^2$.

a) What is the best reply of bidder 1 in a second price auction?

b) What is the best reply of bidder 1 in a first price auction?

c) What is the expected payment of a bidder 1 with valuation x in this situation in a first price auction?

d) What is the ex-ante expected payment of a bidder 1 in this situation in a first price auction?

e) What is the expected payment of a bidder

1 with valuation x in this situation in a second price auction?

f) What is the ex-ante expected payment of a bidder 1 in this situation in a second price auction?

g) Is it surprising that the expected payments in the first and second price auctions here are not the same? Why does the revenue equivalence theorem not apply?

2. Consider an auction with two bidders. Valuations are independently distributed and follow the density function

$$\begin{cases} \sqrt{x} & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

a) What is the equilibrium bidding strategy $\beta^{II}(x)$ in a second price auction?

b) What is the expected payment $m^{II}(x)$ of a bidder with valuation x in the second price auction?

c) Use the revenue-equivalence principle to determine the equilibrium bidding strategy $\beta^{I}(x)$ in the first-price auction.

d) Use direct profit-maximisation (without using revenue-equivalence) to determine the equilibrium bidding strategy $\beta^{I}(x)$ in the first-price auction.

e) What is the equilibrium bidding strategy $\beta^{A}(x)$ in an all-pay auction?

f) Can the seller increase the revenue from the first price auction with the help of a reserve price? If so, what it the optimal reserve price?

g) Now the seller is allowed to set an entry fee. Can such a fee be used to raise the seller's profits in the first price auction? What is the optimal entry fee?

6 Exam, 22. 7. 2008

The exam was set at the University of Jena. Participants had 60 minutes to answer all questions. The exam sheet also contained the formulae from Section 1.

Consider an auction with two bidders. Valuations x are independently distributed and follow the density function

$$f(x) = \begin{cases} 2-2x & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Assume risk-neutral preferences and symmetric equilibria. Please explain all your answers!

1. What is the equilibrium bidding strategy $\beta^{II}(x)$ in the second-price auction?

2. What is the expected payment $m^{II}(x)$ of a bidder with valuation x in the second price auction?

3. Use the revenue-equivalence principle to determine the equilibrium bidding strategy $\beta^{I}(x)$ in the first-price auction.

4. Use direct profit-maximisation (without using revenue-equivalence) to determine the equilibrium bidding strategy $\beta^{I}(x)$ in the first-price auction.

5. What is the equilibrium bidding strategy $\beta^{A}(x)$ in an all-pay auction?

6. Assume that the seller sets a fixed reserve price r. What is the expected payment $m^{I}(x,r)$ of a bidder with valuation x in a first-price auction?

7. What is the expected revenue of the seller in this case?

8. What is the optimal reserve price in a second-price auction?

7 Exam, 5. 2. 2007

The exam was set at the University of Jena. Participants had 60 minutes to answer all questions. The exam sheet also contained the formulae from Section 1.

1. In an auction with three bidders and private valuations these valuations are independent and uniformly distributed over the interval $[0, 3]$.
   a) What is the symmetric equilibrium bidding function in a first-price auction?
   b) What is the symmetric equilibrium bidding function in a second price auction?
   c) What is the symmetric equilibrium bidding function in an all-pay auction?
2. In an auction with three bidders and private valuations these valuations are distributed independently of each other according to $F(x) = x^2$.
   a) What is the symmetric equilibrium bidding function in a first-price auction?
   b) What is the symmetric equilibrium bidding function is a second-price auction?
   c) What is the symmetric equilibrium bidding function in an all-pay auction?
3. In the lecture we derived the symmetric equilibrium bidding function for the first price auction for the case of two bidders and valuations which follow an exponential distribution (i.e. $F(x) = 1 - e^{-\lambda x}$. The result was $\beta^I(x) = \frac{1}{\lambda} - \frac{x}{e^{\lambda x}-1}$.

   Use this result to derive a symmetric bidding function for the all-pay auction.
4. Consider an auction with two bidders and valuations which are distributed independently and uniformly over the interval $[0, 1]$.
   a) Bidder 1 knows that bidder 2 bids according to $\beta_2 = \frac{x}{2}$.
      What is a best reply of bidder 1 in a first price auction? What is the best reply of bidder 1 in a second price auction?
   b) Bidder 1 knows that bidder 2 bids according to $\beta_2 = \frac{1}{2} - \frac{x}{2}$.
      What is a best reply of bidder 1 in a first price auction? What is the best reply of bidder 1 in a second price auction?
   c) Bidder 1 knows that bidder 2 bids according to $\beta_2 = x - \frac{1}{2}$.
      What is a best reply of bidder 1 in a first price auction? What is the best reply of bidder 1 in a second price auction?
   d) Bidder 1 knows that bidder 2 bids according to $\beta_2 = \frac{x}{4}$.
      What is a best reply of bidder 1 in a first price auction? What is the best reply of bidder 1 in a second price auction?
   e) Bidder 1 knows that bidder 2 bids according to $\beta_2 = x$.
      What is a best reply of bidder 1 in a first price auction? What is the best reply of bidder 1 in a second price auction?

8 Exam, 13. 5. 2005

The exam was set at St. Andrews University as a part of EC 4203. Participants had 30 minutes to answer all questions. The exam sheet also contained the formulae from Section 1.

1. An important theorem in auction theory is the revenue equivalence theorem. On which assumptions is the theorem based and what does it say? (25%)
2. Consider the situation of two bidders who have private valuations x_i for one good. These x_i are distributed independently and uniformly over $[0, 1]$. In the lecture we have found that for second-price auctions the weakly dominant bidding strategy for each bidder is to bid simply the own valuation, i.e. $\beta^{II}(x) = x$. Let us call the valuation of the other bidder y, then the expected payment of a bidder with valuation x is in the second-price auction $m^{II}(x) = \int_0^x y \, dy = x^2/2$. Now consider a different auction institution, not a second-price auction, but a first-price all-pay auction, i.e. all bidders pay their bid, but only the highest bidder wins the object.

   Use the revenue equivalence theorem to derive from the equilibrium of the second-price auction the symmetric equilibrium bidding function $\beta^{AP}(x)$ for first-price all-pay auction. (25%)
3. Give an intuition for the curvature of the bidding function in the first-price all-pay auction. (25%)
4. An all-pay auction looks like a strange institution. Where do we find all-pay auctions in real life? (25%)

9 Exam, 14. 10. 2004

The exam was set at Mannheim University. Participants had 90 minutes to answer all questions. The exam sheet also contained the formulae from Section 1.

Explain all your answers in a clear, concise and readable way. Make clear which of your answers belongs to which question. Make also clear what is an answer and what is only an intermediate result. If you can not find an answer for some of the questions in the given time, explain clearly and briefly how you would proceed if you had more time. If you come to the conclusion that in a given case an equilibrium bidding function does not exist, explain why it does not exist.

Consider an auction with two bidders, bidder 1 and bidder 2. Both bidders submit simultaneously their bids b_1 and b_2. Bidder 1 wins if $b_1 > b_2$. Bidder 2 wins if $b_2 > b_1$. If $b_1 = b_2$ the winner is determined by tossing a coin.

1. Consider an All-pay auction. Bidder 1 pays only one half of his bid $b_1/2$. Bidder 2 pays the complete bid b_2.
   a) Valuations X_1 and X_2 for both bidders are distributed according to the function $F(x)$. Can one use the revenue equivalence priciple to determine equilibrium bidding functions?
   b) Determine the equilibrium bidding functions under the assumption that valuations X_1 and X_2 are distributed uniformly over $[0, 1]$.
   c) Now bidders have the same valuation X^*. What are they doing in equilibrium?

2. Now consider the following variant of a first price auction. Bidder 1 pays, if he wins, one half of his bid $b_1/2$. Bidder 2 pays, if he wins, his bid b_2.
   a) Valuations X_1 and X_2 for both bidders are distributed according to the function $F(x)$. Can one use the revenue equivalence priciple to determine equilibrium bidding functions?
   b) Determine the equilibrium bidding functions under the assumption that valuations X_1 and X_2 are distributed uniformly over $[0, 1]$.
   c) Now bidders have the same valuation X^*. What are they doing in equilibrium?

3. Now consider the following variant of a second price auction. Bidder 1 pays, if he wins, $b_2/2$, bidder 2 pays, if he wins, b_1.
   a) The situation is similar to a second price auction. Is it a dominant strategy to bid his true valuation X? If not, is there any other dominant strategy?
   b) Valuations X_1 and X_2 for both bidders are distributed according to the function $F(x)$. Can one use the revenue equivalence priciple to determine equilibrium bidding functions?
   c) Determine the equilibrium bidding functions under the assumption that valuations X_1 and X_2 are distributed uniformly over $[0, 1]$.
   d) Now bidders have the same valuation X^*. What are they doing in equilibrium?

4. Which of the above three auction types would you choose — given the above assumptions — if you are the seller?
   a) Assume first a risk-neutral seller.
   b) Now assume a risk-averse seller. How can you model risk-aversion? Can you find a specific functional form of a utility function to determine the utility maximising auction under risk aversion. Can you make a more general statement without assuming a specific function form?

5. Consider the following situation. Two bidders have valuations X_1 and X_2 for an object. Valuations are distributed according to $F(x)$. Bidders participate sequentially in the auction. First the auction determines a buy-now price p. This price is known to both bidders. Then bidder 1 submits his bid b_1. Bidder 2 does not observe b_1. If $b_1 \geqslant p$ then bidder 1 obtains the object and the auction ends. Otherwise bidder 2 can submit his bid b_2 and the bidder with the highest bid wins. Assume first the the winner pays the second highest bid.
   a) What is an equilibrium strategy of bidder 2
   b) What is an equilibrium strategy of bidder 1

c) What buy-now price is chosen by the revenue maximising auctioneer.

6. Now assume that the winner pays the own bid.

    a) What is an equilibrium strategy of bidder 2

    b) What is an equilibrium strategy of bidder 1

    c) What buy-now price is chosen by the revenue maximising auctioneer.

10 Exam, 27. 7. 2004

The exam was set at Mannheim University. Participants had 90 minutes to answer all questions. The exam sheet also contained the formulae from Section 1.

Explain all your answers in a clear, concise and readable way. Make clear which of your answers belongs to which question. Make also clear what is an answer and what is only an intermediate result. If you can not find an answer for some of the questions in the given time, explain clearly and briefly how you would proceed if you had more time. If you come to the conclusion that in a given case an equilibrium bidding function does not exist, explain why it does not exist.

1. Consider an auction with an entry fee. Assume that the seller has a valuation of zero for the object.

    a) Can it be good for the seller to charge an entry fee? Give a brief intuition? Which effects does one have to consider?

    b) Is the resulting allocation in a standard auction with private values with an entry fee always efficient? Explain!

    c) What other problem do buyers and sellers face in such a situation?

2. Two bidders have private valuations X_1 and X_2.

    a) Both valuations are distributed independently and uniformly over $[0, 1]$. What is an equilibrium bidding function in a first price auction?

    b) What is in this situation an equilibrium bidding function in a second price auction?

    c) Now the valuation of bidder 2 is distributed uniformly over $[0, 2]$. What is now an equilibrium bidding function in a second price auction?

    d) What is an equilibrium bidding function in a first price auction in this case?

3. Consider an auction with two bidders and interdependent valuations. Bidder 1 obtains a private signal x_1 and Bidder 2 obtains a private signal x_2. The two signals are independently distributed. The valuation for bidder 1 is

$$v_1 = (1 - \alpha)x_1 + \alpha x_2$$

and the valuation for bidder 2 is

$$v_2 = (1 - \alpha)x_2 + \alpha x_1 \,.$$

    a) Assume that bidders follow a symmetric equilibrium bidding strategy. For what values of α can no efficient allocation be obtained? How does the result depend on the distribution of each signal. Explain!

    b) Assume that signals are distributed independently and uniformly over $[0, 1]$. What is an equilibrium bidding strategy in a second price auction?

    c) What is an equilibrium bidding strategy in an English auction? (model the English auction as we did in the lecture)

4. Consider the following variant of an all-pay auction. There are two bidders, 1 and 2, with private valuations X_1 and X_2 for an object. They make bids b_1 and b_2 respectively. The bidder with the highest bid obtains the object. Regardless of who is the winner of the auction bidder 1 always pays $\frac{1}{2}b_1$ and bidder 2 always pays b_2.

    a) Can one use the revenue-equivalence principle to determine an equilibrium bidding function?

    b) If not, why not?

    c) If so, how could one proceed.

    d) Find an equilibrium.

5. Consider the following second-price all-pay auction. There are two bidders, 1 and 2 who make bids b_1 and b_2 respectively. The bidder with the highest bid obtains a fixed price P. Regardless of who is the winner of the auction,

both bidders pay a cost which is characterised by the smaller bid $\underline{b} = \min\{b_1, b_2\}$ and individual parameters for each bidder, θ_1 and θ_2. Bidder 1 pays $c_1 = \underline{b} \cdot \theta_1$, bidder 2 pays $c_2 = \underline{b} \cdot \theta_2$.

a) Assume that both bidders know that $\theta_1 = \theta_2$. What can you say about equilibrium bidding strategies?

b) Assume that $\theta_1 = \frac{1}{2}\theta_2$. What can you say about equilibrium bidding strategies?

c) Assume that $\theta_1 = 1$ and θ_2 is uniformly distributed over $[\frac{1}{2}, \frac{3}{2}]$. What can you now say about equilibrium bidding strategies?

d) Assume that both θ_1 and θ_2 are uniformly distributed over $[\frac{1}{2}, \frac{3}{2}]$. What can you now say about equilibrium bidding strategies?